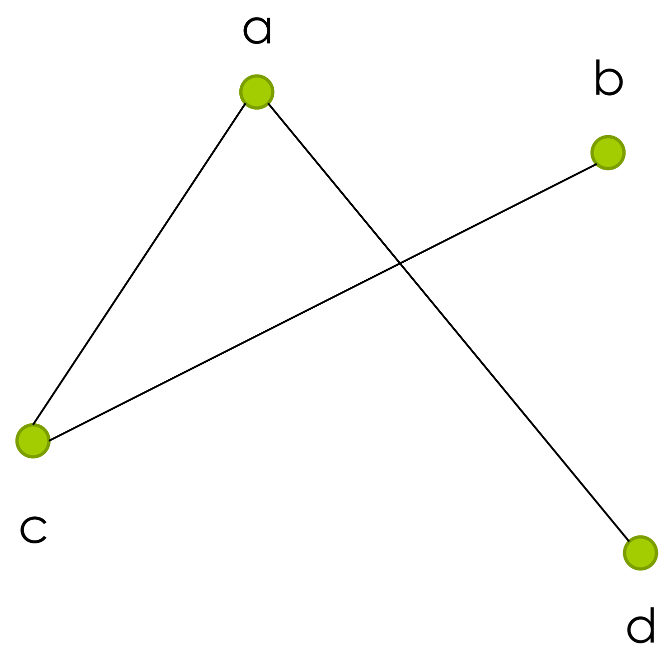
Properties of k-Residue

By Randy R. Davila, David Amos, and Dr. Ryan Pepper.

Basic Graph Theory Definitions

- A <u>*simple graph*</u> G is a pair G = (V,E) where V is a finite nonempty set, called the vertices of G, and E is a subset of V X V (i.e., a set E of two-element subsets of V), called the edges of G, where we do not consider loops or multi-edges.

Example:



$G = (V, E)$

V = {a,b,c,d}

E = {(a,c), (a,d), (c,b)}

Basic Graph Theory Definitions

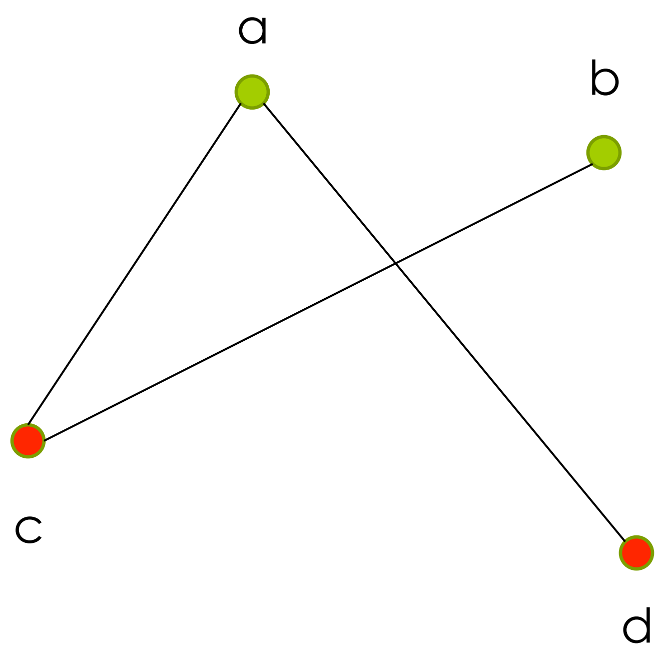
- Let $A \subseteq V(G)$. The *subgraph induced by* A is the graph with vertex set A and edge set $E \downarrow A = \{xy \in E(G) : x, y \in A\}$.

- Let $v \in V(G)$. The graph $G - \{v\}$ is obtained by removing v from $V(G)$ and all edges incident to v from $E(G)$.

Basic Graph Theory Definitions

- Two vertices of a graph G are said to be *independent* if there does not exist an edge between them. A collection of independent vertices forms an independent set. The cardinality of the largest such set in G, is the *independence number* and is denoted by $\alpha(G)$.

Example:



$G=(V,E),\ \alpha(G)=2$

V = {a,b,c,d}

E = {(a,c), (a,d), (c,b)}

Degree Sequence Definitions

- The number of vertices adjacent to some vertex v, is known as the *degree of v and is* denoted by $d(v)$. The maximum and minimum of these degrees is denoted by $\Delta(G)$ and $\delta(G)$ respectively.
- The *degree sequence D* of a graph G is the non-increasing sequence of the degrees of G. G is said to realize D.

Havel-Hakimi Derivatives

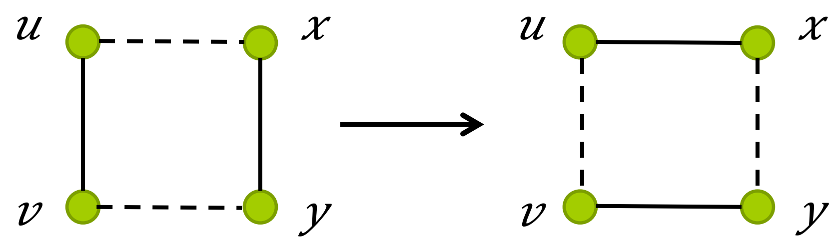
- Let $D=\{d_1, d_2, \ldots, d_n\}$ be a non-increasing sequence. Deleting d_1 and reducing the next d_1 many terms by one is known as the *Havel-Hakimi derivative* of D, denoted D'.
- $D=\{d_1, d_2, \ldots, d_n\}$
  $D' =\{d_2 -1, d_3 -1, \ldots, d_{d_1 +1} -1, d_{d_1 +2}, \ldots, d_n\}$
- We define the *Havel-Hakimi process* as the process of iteratively taking multiple Havel-Hakimi derivatives. $D^{(i)}$ will denote taking up to the $i-th$ derivative of D.

Havel-Hakimi Theorems

- Havel [5] and Hakimi [4] demonstrated independently the following theorems.
- **Theorem 1 (Havel-Hakimi):** A sequence D is graphic if and only if its Havel-Hakimi derivative $D\uparrow'$ is graphic.
- **Theorem 2 (Havel-Hakimi)**: A sequence $D=\{d\downarrow 1, d\downarrow 2, \ldots, d\downarrow n\}$ is graphic if and only if the sequence terminates with a list of zero's for some dervative $i=1,2,\ldots,n-1$.

Ryser Switches

- Let $u,v,x,y \in V(G)$ be four different vertices such that $uv, xy \in E(G)$ and $ux, vy \notin E(G)$. A Ryser switch is performed by removing the edges uv and xy and replacing them with ux and vy.



Application of Ryser Switches

- Ryser switches can be used to prove the Havel-Hakimi theorems (see proof in West)
- The key idea is that every graph G can be transformed via a (possibly empty) sequence of Ryser switches into a graph H such that $D(G)=D(H)$ and H has a maximum degree vertex v such that $H-\{v\}$ realizes $D'(G)$.

Havel-Hakimi Residue

- In 1988 Fajtlowicz [1] introduced the Residue of a graph as the number of zero's produced by the terminating Havel-Hakimi process. Residue will be denoted by $R(G)$.
- Fajtlowicz computer program "Graffiti" conjectured that $R(D(G)) \leq \alpha(G)$ during the same time period.
- A few years later, Favaron et. Al. [2] supplied a proof. In 1994 a simpler proof was provided by Griggs and Kleitman [3].

Majorization

- Central to both mentioned proofs is the idea of *majorization*.
- Given two sequences $A=\{a_i\}$ and $B=\{b_i\}$, A is said to *majorize* B if the following conditions hold:

1. A and B have the same length n.
2. $\sum_{i=1}^{n} a_i = \sum_{i=1}^{n} b_i$
3. $\sum_{i=1}^{k} a_k \geq \sum_{i=1}^{k} b_i$

If A *majorizes* B we write $A \succcurlyeq B$.

Majorization Lemma

- **Lemma 3 (Favaron, Mahéo, and Saclé)**: Let D_1 and D_2 be graphic sequences of length n. If $D_1 \succcurlyeq D_2$, then $R(D_1) \geq R(D_2)$.

k-Residue Origins

- The elimination sequence introduced by Triesch [7], denoted by $E(D)$, is the sequence of integers eliminated at each step of the Havel-Hakimi process, together with the remaining zero's upon termination.
- Jelen [6] later defined a generalization of the residue of a sequence D, called the k-Residue and denoted $R\downarrow k\,(D)$, based on Triesch's elimination sequence.

K-Residue and k-Indepedence Definitions

- Let D be a graphic sequence and let $E(D)$ be the elimination sequence of D. The <u>*k-residue*</u> of D is given by the sum

$$R_k(D)=\frac{1}{k}\sum_{i=0}^{k-1}(k-i)f_i(E(D)),$$

  where $f_i(E(D))$ is the frequency of i in $E(D)$.

- The <u>*k-independence number*</u> of a graph G, written $\alpha_k(G)$, is the cardinality of a largest subset S, of vertices for which the maximum degree of the subgraph induced by S is at most $k-1$.

Jelen's Contributions and k-Indepedence

- **Theorem 4 (Jelen).** Let D_1 and D_2 be graphic sequences such that $D_1 \succcurlyeq D_2$, then $R_k(D_1) \geq R_k(D_2)$.
- **Theorem 5 (Jelen).** For any graph G, $R_k \leq \alpha_k$.

Results

- **Proposition 6.** *If* $k \leq n$*, then* $R_k(K_n) = \frac{k+1}{2}$.

Results

- **Proposition 7.** *Let $\Delta(G)$ be the maximum degree of an n-vertex graph G with m edges. If $k \geq \Delta(G)$, then $R{\downarrow}k\ (G) = n - m/k$*

Results

- **Theorem 7.** *For any two disjoint graphs* G *and* H, $R{\downarrow}k\,(G \cup H) \leq R{\downarrow}k\,(G) + R{\downarrow}k\,(H)$.

Results

- **Corollary 8.** *For any disconnected graph* G with p components G_i, $R_k(G) \leq \sum_{i=1}^{p} R_k(G_i)$.

Results

- **Theorem 9.** *For two graphs A and B, $R{\downarrow}k(A \cup B) = R{\downarrow}k(A) + R{\downarrow}k(B)$ for every k if $E(A \cup B) = E(A) + E(B)$.*

Results

- **Theorem 10.** If a graph G has a maximum degree cut vertex whose removal splits G into $p \geq 2$ components G_i , $i = 1, 2, \ldots, p$, then $R_k(G) \leq \sum_{i=1}^{p} R_k(G_i)$.

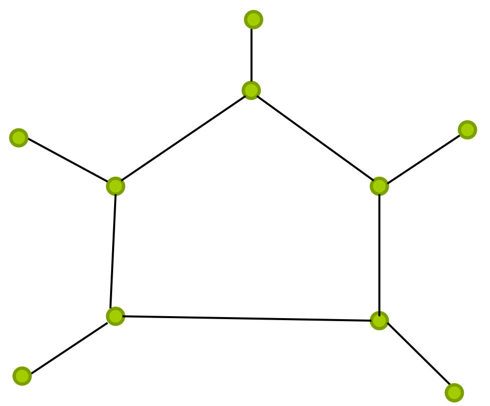
Results

- **Theorem 11.** Let (G,f) be a function graph on G. Then $R_k(G,f) \le 2R_k(G)$.

What else can be done?

Possible Modifications to the Current Residue Definition

- Consider the graph below:



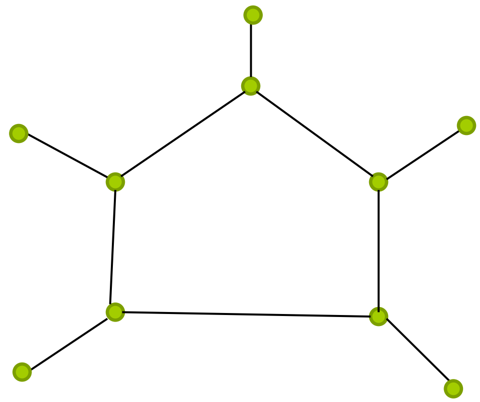
$R(D(G))=4$, $\alpha(G)=5$
- Are there any modifications that can be made to residue to fix the difference?

Possible Solution

- Define the <u>*Special Residue Switch*</u> as follows: subtract 1 from a min degree of $D(G)$ and add 1 to the largest degree of $D(G)$ less than the max degree of $D(G)$. Denote the new sequence obtained by this process $D\uparrow *(G)$.
- This new sequence will have the following properties: $D(G) \prec D\uparrow *(G)$ implying $R(D(G)) \leq R(D\uparrow *(G))$

Using the Special Residue Switch

- Consider the graph below:



$R(G)=4$, $\alpha(G)=5$
$R(D^{*}(G))=5$

What else?

- Determine the probability that residue = independence number.
- Find relations of residue to other graph invariants.
- Characterize what operations on a graph change the residue of the graph.